Trilinear Embedding Theorem for Elliptic Partial Differential Operators in Divergence Form with Complex Coefficients

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Elliptic partial differential operators (of 2nd order)

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Generalizations:

- complex A;
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- higher dimensions.

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For $p \in [1, \infty]$ set

$$\Delta_p(A) = \operatorname*{ess\,inf}_{x \in \Omega} \min_{|\xi|=1} \Re \left\langle A(x)\xi, \xi + |1 - 2/p|\bar{\xi} \right\rangle_{\mathbb{C}^n}$$

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Evans: Partial Differential Equations (AMS 2010), p. 327

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p-**ellipticity**: may be of interest for the L^p theory of elliptic PDE.

Examples

(i) convexity of power functions (Bellman functions),

(ii) **dimension-free bilinear embeddings**,

(iii) L^p-contractivity of semigroups,

(iv) holomorphic functional calculus,

(v) square function estimates,

(vi) trilinear embeddings and Kato–Ponce inequalities (with Kovač and Škreb).

(vii) regularity theory of elliptic PDE with complex coefficients (Dindoš-Pipher),

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Obvious:

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For any $A \in \mathcal{A}(\Omega)$ set

$$\mu(A) := \operatorname{ess\,inf} \Re \frac{\langle A(x)\xi, \xi\rangle}{|\left\langle A(x)\xi, \bar{\xi}\right\rangle|};$$

ess inf over all $x \in \mathbb{R}^n$ and all $\xi \in \mathbb{C}^n$ for which $\left\langle A(x)\xi, \bar{\xi}\right\rangle \neq 0$.
The key assumption $\Delta_p(A) > 0$ is equivalent to

$$|1 - 2/p| < \mu(A)$$

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The key assumption $\Delta_p(A) > 0$ is equivalent to

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Furthermore: $\Delta_p(A)$ is Lipschitz continuous in both p and A.

The Dindoš–Pipher condition (2016)

For some $\varepsilon = \varepsilon(A, p) > 0$ and almost all $x \in \Omega$,

$$\begin{aligned}\langle \Re A(x)\lambda, \lambda\rangle_{\mathbb{R}^d} &+ \langle \Re A(x)\eta, \eta\rangle_{\mathbb{R}^d} \\ &+ \left\langle \left(\sqrt{p'/p}\,\Im A(x) - \sqrt{p/p'}\,\Im A(x)^T\right)\lambda, \eta \right\rangle_{\mathbb{R}^d} \\ &\geqslant \varepsilon\left(|\lambda|^2 + |\eta|^2\right)\end{aligned}$$

for all $\lambda, \eta \in \mathbb{R}^d$. Here $p' = p/(p-1)$ is the conjugate exponent of p.

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The Dindoš–Pipher condition was derived as a strengthening of a condition by Cialdea–Maz'ya (2005).

p-ellipticity comes from studying (generalized) convexity properties of **power functions** of a single complex variable.

Study of power functions was motivated by our attempts to understand convexity of a particular **Bellman function** due to Nazarov and Treil, which comprises tensor products of power functions.

This was in turn pursued as a part of our (D.–Volberg 2011, Carbonaro–D. 2015) efforts to prove **bilinear embedding** theorem for arbitrary complex accretive matrices A.

Bilinear embedding theorem for divergence-form operators

Define
$$L_A u := -\mathrm{div}\,(A\nabla u).$$
Operator semigroups: $\varphi(t) = e^{-t\mathcal{L}} f$ solves $\varphi' + \mathcal{L}\varphi = 0$, $\varphi(0) = f$.

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Theorem (Carbonaro–D. 2015)

For $p > 1$, $q = p/(p-1)$, $A, B \in \mathcal{A}_p(\mathbb{R}^n)$, $f, g \in C_c^\infty(\mathbb{R}^n)$ we have

$$\int_0^\infty \int_{\mathbb{R}^n} |\nabla_x e^{-tL_A} f(x)|\,|\nabla_x e^{-tL_B} g(x)|\, dx\, dt \leqslant \frac{20}{\Delta_p} \cdot \frac{\Lambda}{\lambda} \|f\|_p \|g\|_q\,,$$

where $\Delta_p = \min\{\Delta_p(A), \Delta_p(B)\}$ in $\Lambda = \max\{\Lambda(A), \Lambda(B)\}$.

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Motivation: square function estimates (Auscher 2004).
D.–Volberg (2007): $A = B$ **real** and $p \in (1, \infty)$.
The condition $\Delta_p > 0$ is sharp (L^p **contractivity** of the semigroup).

Bilinear embeddings and *heat flows* - main developments

$Ahlfors - Beurling\ operator$	$(Petermichl - Volberg\ 2002$
	$\&\ Nazarov - Volberg\ 2003)$
$A = B\ real$	$(D. - Volberg\ 2008)$
$A, B = e^{i\phi} I$	$(Carbonaro - D.\ 2012)$
$A, B\ arbitrary\ complex$	$(Carbonaro - D.\ 2016)$
$A, B, \Omega\ arbitrary$	$(Carbonaro - D.\ 2018)$

The heat flow method. Proof of the bilinear embedding.

Let $Q : \mathbb{C}^2 \to \mathbb{R}$. Define $\mathcal{E} : [0, \infty) \to \mathbb{R}_+$ by

$$\mathcal{E}(t) = \int_{\mathbb{R}^n} Q\left(e^{-tL_A} f, e^{-tL_B} g\right).$$

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from below and above.

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Upper estimate

Suppose that $0 \leqslant Q(\zeta, \eta) \leqslant \mathfrak{b}_0\left(|\zeta|^p + |\eta|^q\right)$ for $\zeta, \eta \in \mathbb{C}$. Then

$$-\int_0^\infty \mathcal{E}'(t)\,dt \leqslant \mathcal{E}(0) \leqslant \mathfrak{b}_0(\|f\|_p^p + \|g\|_q^q).$$

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Lower estimate

Integration by parts gives, for $h_t = (e^{-tL_A}f, e^{-tL_B}g)$,

$$-\mathcal{E}'(t) = \int_{\mathbb{R}^n} H_Q^{(A,B)}[h_t; \nabla h_t]$$

Here, roughly speaking, the ***generalized Hessian form*** $H_Q^{(A,B)}$ is

$$H_Q^{(A,B)}[v;\omega] = \Re \left\langle d^2 Q(v)\,\omega, \begin{bmatrix} A & \\ & B \end{bmatrix} \omega \right\rangle_{\mathbb{C}^n \times \mathbb{C}^n}$$

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Similarly: for $F : \mathbb{C} \to \mathbb{R}$ denote

$$H_F^A[\zeta;\xi] = \Re \left\langle d^2 F(\zeta)\,\xi, A\xi \right\rangle_{\mathbb{C}^n}.$$

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Suppose additionally that

$$H_Q^{(A,B)}[v;(\alpha,\beta)] \geqslant \mathfrak{a}_0 |\alpha||\beta|$$

for $v \in \mathbb{C}^2$, $\alpha, \beta \in \mathbb{R}^{2n}$. Then

$$\mathfrak{a}_0 \int_0^\infty \left\langle |\nabla e^{-tL_A} f|, |\nabla e^{-tL_B} g| \right\rangle_{L^2(\mathbb{R}^n)} dt \leqslant -\int_0^\infty \mathcal{E}'(t)\, dt\,.$$

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We (almost) get the bilinear embedding:

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Replace $f \to \sigma f$, $g \to \sigma^{-1} g$ and maximize in $\sigma > 0$. □

The heat flow method - SUMMARY

Bilinear embedding reduces to finding a function $Q : \mathbb{C}^2 \to \mathbb{R}$ of order C^2 such that:

(i) the corresponding flow is regular;

(ii) for all $(\zeta, \eta) \in \mathbb{C}^2$:

$$0 \leqslant Q(\zeta, \eta) \lesssim |\zeta|^p + |\eta|^q$$

(iii) for any $v \in \mathbb{C}^2$ and $\alpha, \beta \in \mathbb{R}^{2n}$:

$$H_Q^{(A,B)}[v; (\alpha, \beta)] \gtrsim |\alpha||\beta|.$$

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When $H_Q^{(A,B)}$ is the usual Hessian matrix of Q (the case $A = B \equiv I$), a suitable function Q is **already known to exist** ($\to$ a natural starting point).

The Nazarov–Treil function

Bellman function method: Nazarov–Treil–Volberg 1994

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An early concrete example:

Nazarov–Treil (1995) + simplification D.–Volberg (2008)

Let $p \geqslant 2$, $1/p + 1/q = 1$ and $\delta > 0$.

Define the *Bellman function* $Q = Q_{p,\delta} : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{R}_+$ as

$$Q(\zeta,\eta) = |\zeta|^p + |\eta|^q + \delta \begin{cases} |\zeta|^2 |\eta|^{2-q} & ; \quad |\zeta|^p \leqslant |\eta|^q \\ \dfrac{2}{p}\,|\zeta|^p + \left(\dfrac{2}{q} - 1\right)|\eta|^q & ; \quad |\zeta|^p \geqslant |\eta|^q \,. \end{cases}$$

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Structural feature: tensor products of **power functions**.

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$$Q = c_1|\zeta|^p + c_2|\eta|^q.$$

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Hence

$$(A \oplus B)^T d^2 Q = \begin{bmatrix} c_1 A^T d^2|\zeta|^p & 0 \\ 0 & c_2 B^T d^2|\eta|^q \end{bmatrix}.$$

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$$Q = c_1|\zeta|^p + c_2|\eta|^q.$$

Hence

$$(A \oplus B)^T d^2 Q = \begin{bmatrix} c_1 A^T d^2|\zeta|^p & 0 \\ 0 & c_2 B^T d^2|\eta|^q \end{bmatrix}.$$

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Problem: in the **bad region** $\{|\zeta|^p \leqslant |\eta|^q\}$ we have

$$Q = |\zeta|^p + |\eta|^q + c_3|\zeta|^2|\eta|^{2-q}.$$

More complicated Hessian of Q.

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Power functions

For $r \geqslant 0$ define $F_r(\zeta) = |\zeta|^r$, $\zeta \in \mathbb{C}$.

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$$\Delta_p(A) = \frac{2}{p^2} \operatorname*{ess\,inf}_{x\in\Omega} \min_{|\xi|=1} \min_{|\zeta|=1} H_{F_p}^{A(x)}[\zeta;\xi]\,.$$

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Equivalence of the above?

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Equivalence of the above?

Recall: $\Delta_p(A)$ is a sesquilinear form generated by A and the ***Hessian matrix of the power function*** $F_p(\zeta) = |\zeta|^p$

Hessian of the power function $F_p(x_1, x_2) = \left(x_1^2 + x_2^2\right)^{p/2}$

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How to calculate the 2×2 Hessian matrix of F_p?

For $\varsigma \in \mathbb{C}^n$ set $\mathfrak{I}_p \varsigma = \varsigma + (1 - 2/p)\bar{\varsigma}$.

Theorem

For any $\zeta \in \mathbb{C}$ *and* $\xi \in \mathbb{C}^n$ *we have*

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Generalized convexity of F_p implies gen. conv. of Q

Theorem

If $p \geqslant 2$ *and* $A, B \in \mathcal{A}_p(\Omega) > 0$ *then, a.e.* Ω,

$$H_Q^{(A,B)}[v; (\omega_1, \omega_2)] \gtrsim |\omega_1||\omega_2| \,,$$

for some $\delta = \delta(\Delta_p, \lambda, \Lambda) \in (0, 1)$ *and* $Q = Q_{p,\delta}$ *as above.*

From L^p contractivity to p-ellipticity (a shortcut)

- $(e^{-tL})_{t>0}$ is L^p-contractive on a Banach space X

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Exact proof based on Nittka (2012).

Example: Spectral multipliers for generators of symmetric contraction semigroups

Origin of the theory: Stein 1970

(Ω, ν): σ-finite measure space

$\mathcal{A}$: a nonnegative self-adjoint operator on $L^2(\Omega, \nu)$.

For $t > 0$ define

$$P_t := e^{-t\mathcal{A}}$$

We assume that $(P_t)_{t>0}$ is symmetric contraction semigroup:

for all $t > 0$ and all $p \in [1, \infty]$,

$$\|P_t f\|_p \leqslant \|f\|_p \qquad \forall f \in L^p(\Omega, \nu) \cap L^2(\Omega, \nu).$$

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$$\mathcal{A} = \int_0^\infty \lambda \, dE(\lambda) \, .$$

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C. Fefferman (1971): for $n > 1$ we have

$$\chi_{(-1,1)} \in \mathcal{M}_p(-\Delta_n) \qquad \Longleftrightarrow \qquad p = 2$$

Multiplier theorem (Carbonaro–D. 2012)

QUESTION: the smallest angle ϑ for which there is Hörmander-type holomorphic functional calculus on L^p in the sector $\mathbf{S}_\vartheta$ for all generators of symmetric contraction semigroups?

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Calculation of $\Delta_p(e^{i\phi}I)$

$$\Delta_p(e^{i\phi}I) = \min_{|\xi|=1} \Re \left\langle e^{i\phi}\xi, \xi + |1-2/p|\bar{\xi} \right\rangle_{\mathbb{C}^n}$$

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Trilinear embedding for complex elliptic operators

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Theorem (Carbonaro – D. – Kovač – Škreb 2020)

Let $p, q, r \in (1, \infty)$ *satisfy* $1/p + 1/q + 1/r = 1$. *Assume that the matrices* $A, B, C : \Omega^n \to \mathbb{C}$ *are* $\max\{p, q, r\}$*-elliptic. Let* L_A *be the elliptic operator* $u \mapsto -div(A\nabla u)$ *subject to either Dirichlet, Neumann or mixed boundary conditions on* Ω. *Then*

$$\int_0^\infty \int_\Omega |\nabla e^{-tL_A} f||\nabla e^{-tL_B} g||e^{-tL_C} h| \, dx \, dt \lesssim \|f\|_p \|g\|_q \|h\|_r .$$

When $\Omega = \mathbb{R}^n$, *the theorem holds under weaker conditions, i.e.:*

- A *is* p*-elliptic and* $(1 + p/q)$*-elliptic*
- B *is* q*-elliptic and* $(1 + q/p)$*-elliptic*
- C *is* r*-elliptic.*

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- Trilinear embedding $\Rightarrow$ bilinear embedding.
- No more symmetry with respect to the conjugation of p.
- Proof: heat flow + Bellman function. We need three variables.

The Kovač-Škreb function (2018)

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Write $q/r = 1 + 2\varepsilon$ and define

$$\mathfrak{X}(u,v,w) :=$$

$$\begin{cases} D|u|^p + |v|^q + E|w|^r; & |u|^p \leqslant |w|^r \leqslant |v|^q, \\ \left(D - \frac{E}{p-1}\right)|u|^p + |v|^q + \frac{Ep}{p-1}|u||w|^{r-r/p}; & |w|^r \leqslant |u|^p \leqslant |v|^q, \\ \left(D - \frac{E+1}{p-1}\right)|u|^p + \frac{p}{p-1}|u||v|^{q-q/p} + \frac{Ep}{p-1}|u||w|^{r-r/p}; & |w|^r \leqslant |v|^q \leqslant |u|^p, \\ \left(D - \frac{E+1}{p-1}\right)|u|^p + \frac{q}{2}|u||v|^2|w|^{1-r/q} + (E-\varepsilon)\frac{p}{p-1}|u||w|^{r-r/p}; & |v|^q \leqslant |w|^r \leqslant |u|^p, \\ \left(D - \frac{q}{2p}\right)|u|^p + \frac{q^2}{2p(q-2)}|u|^{p-2p/q}|v|^2 + \frac{\varepsilon q}{q-2}|v|^2|w|^{r-2r/q} & \\ \qquad + (E-\varepsilon)|w|^r; & |v|^q \leqslant |u|^p \leqslant |w|^r, \\ D|u|^p + \frac{q}{p(q-2)}|v|^q + \frac{\varepsilon q}{q-2}|v|^2|w|^{r-2r/q} + (E-\varepsilon)|w|^r; & |u|^p \leqslant |v|^q \leqslant |w|^r. \end{cases}$$

Generalized convexity of the Kovač-Škreb function

Theorem (Carbonaro – D. – Kovač – Škreb 2020)

Under the conditions on p, q, r, A, B, C, specified in the trilinear embedding theorem, there exist $D, E > 0$ such that $\mathfrak{X}$ satisfies:

a) *for all $u, v, w \in \mathbb{C}$,*

$$\mathfrak{X}(u, v, w) \lesssim |u|^p + |v|^q + |w|^r;$$

b) *for almost every $x \in \Omega$,*

$$H_{\mathfrak{X}}^{(A,B,C)(x)}[(u, v, w); (\zeta, \eta, \xi)] \gtrsim |w||\zeta||\eta|$$

for all $(u, v, w) \in \mathbb{C}^3 \setminus \Upsilon$ and $(\zeta, \eta, \xi) \in (\mathbb{C}^d)^3$.

The implied constants depend on p, q, r and $*$-ellipticity constants of A, B, C alluded to in the theorem's assumptions.

The Kato–Ponce inequality for complex elliptic operators

Theorem (Carbonaro – D. – Kovač – Škreb 2021)

Let $p_1, q_1, p_2, q_2, \wp \in (1, \infty)$ *be such that*

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If A *is* $\max\{p_1, p_2, q_1, q_2, \wp'\}$*-elliptic and* $\beta \in (0, 1/\wp)$*, then*

$$\left\| L_A^\beta (fg) \right\|_\wp \lesssim \left\| L_A^\beta f \right\|_{p_1} \|g\|_{q_1} + \|f\|_{p_2} \left\| L_A^\beta g \right\|_{q_2}.$$

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- Interpretation of L_A^β: functional calculus for *sectorial operators*.
- We do not include the case $p_2, q_1 = \infty$.

The Kato–Ponce inequality for complex elliptic operators

Theorem (Carbonaro – D. – Kovač – Škreb 2021)

Let $p_1, q_1, p_2, q_2, \wp \in (1, \infty)$ *be such that*

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{\wp}.$$

If A *is* $\max\{p_1, p_2, q_1, q_2, \wp'\}$*-elliptic and* $\beta \in (0, 1/\wp)$*, then*

$$\left\| L_A^\beta (fg) \right\|_{\wp} \lesssim \left\| L_A^\beta f \right\|_{p_1} \|g\|_{q_1} + \|f\|_{p_2} \left\| L_A^\beta g \right\|_{q_2}.$$

- Interpretation of L_A^β: functional calculus for *sectorial operators*.
- We do not include the case $p_2, q_1 = \infty$.
- A new approach to Kato–Ponce inequalities.

Elements of the proof of the Kato–Ponce inequality

Elements of the proof:

- Dualization: for $1/\wp + 1/r = 1$ it is enough to prove

$$\left| \int_\Omega fg \overline{L_{A^*}^\beta h} \,\mathrm{d}x \right| \lesssim \left(\left\| L_A^\beta f \right\|_{p_1} \|g\|_{q_1} + \|f\|_{p_2} \left\| L_A^\beta g \right\|_{q_2} \right) \|h\|_r .$$

- Splitting the left-hand side by *Calderón reproducing formula*:

$$\int_\Omega fg \overline{L_{A^*}^\beta h} \,\mathrm{d}x \\ = - \int_0^\infty \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t} \left(\phi_\alpha(tL_A)f \cdot \phi_\alpha(tL_A)g \cdot \overline{\phi_\alpha(tL_{A^*})L_{A^*}^\beta h} \right) dx\, dt,$$

where $\psi_\alpha(z) = z^\alpha e^{-z}$ and

$$\phi_\alpha(z) = \frac{1}{\Gamma(\alpha)} \int_1^\infty \psi_\alpha(sz) \frac{ds}{s} .$$

Elements of the proof of the Kato–Ponce inequality

- By

  (i) differentiating the right-hand side above,

  (ii) splitting/decomposing further,

  (iii) integrating by parts (definition of L_A)

  we eventually arrive at the crucial term to be estimated:

$$J_3' = \int_0^\infty \int_\Omega \overline{\psi_{\alpha-1}(tL_{A^*})L_{A^*}^\beta h} \int_t^\infty \left\langle (A + A^T)\nabla L_A \psi_{\alpha-1}(sL_A)f, \nabla\overline{\phi_\alpha(sL_A)g} \right\rangle \mathrm{d}s\, \mathrm{d}x\, \mathrm{d}t.$$

Elements of the proof of the Kato–Ponce inequality

- By *subordination to imaginary powers of L_A* (via the inversion formula for the *Mellin transform*) we end up estimating

$$\int_1^\infty s^{\beta-1} \left(\int_0^\infty \int_\Omega \left| \nabla e^{-stL_A} L_A^{iu_1} L_A^{\beta} f \right| \left| \nabla e^{-stL_A} L_A^{iu_2} g \right| \left| e^{-tL_{A^*}} L_{A^*}^{iu_3} h \right| \,\mathrm{d}x\,\mathrm{d}t \right) \mathrm{d}s.$$

  for $u_{1,2,3} \in \mathbb{R}$.

- Thus the proof will be finished once we obtain
  (i) trilinear embedding with **adequate control of the embedding constants for** (sA, sB, C) **in terms of** $s > 0$, and
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We have been profusely using the analiticity of e^{-tL_A} and holomorphic functional calculus for L_A.

Thank you for your attention